WEAK MATRIX ELEMENT CALCULATIONS ON THE LATTICE USING STAGGERED FERMIONS

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We outline a lattice method, using staggered fermions, for the evaluation of matrix elements relevant for weak decays of strange mesons. We require three inversions per configuration.

1. Introduction. The recent interest in the numerical lattice evaluation of matrix elements relevant to low-energy weak hadronic interactions has focused on formulating the problem in terms of Wilson fermions [1-3]. However, the Wilson fermion formulation does not respect the chiral symmetry which is expected to be approximately valid for the continuum limit, and there are consequent complications [4]. Therefore one might prefer to consider the problem in terms of staggered fermions, where sufficient chiral symmetry is retained to ensure that the matrix elements have chiral behaviour analogous to that in the continuum [5].

Unfortunately, staggered fermions have problems of their own. Firstly, their use entails the use of multi-link operators and, ideally, the generation of propagators from many sources. Both of these points tend to increase the computational effort required. Secondly, in the continuum limit, one staggered fermion field corresponds to four "flavours" of Dirac fermion [6]. However, at non-zero lattice spacing, these "Susskind flavours" mix, and even with degenerate masses we have only a discrete flavour symmetry [5]. Thus, if we identify Susskind flavour with continuum flavour, the formulation of weak hadronic transitions, whose characteristic feature is flavour non-conservation, somewhat non-intuitive.

One way to avoid the second difficulty, as proposed in ref. [5], is the use of three species of staggered fermion field, one for each of the (light) continuum flavours. This leads to a direct correspondence between the chiral symmetry in the continuum and on the lattice. The apparent disadvantage of having four lattice fermions for each continuum flavour should result in nothing more serious than an overcounting, which can be kept track of provided that we are working within the quenched approximation.

The approach outlined in ref. [5] was motivated by the desire to have lattice matrix elements satisfying Ward identities analogous to those in the continuum. In this letter we propose a method for weak matrix elements, again using three species of staggered fermion, in which our prime concern is to use propagators from as few sites as possible (one, with the requirement of two extra inversions), and thus make progress possible with little more effort than is required for hadron mass calculations.

2. Review of aims. To be specific, we consider $K^0 \to \pi^+\pi^-$, $\Delta S=1$ transitions, for which the effective hamiltonian [7] is an expansion over six four-quark operators,

$$\mathcal{O}_1 = (\bar{d}\gamma_{\mu L}s)(\bar{u}\gamma_{\mu L}u) - (\bar{d}\gamma_{\mu L}u)(\bar{u}\gamma_{\mu L}s) ,$$

$$\mathcal{O}_2 = (\bar{d}\gamma_{\mu L}s)(\bar{u}\gamma_{\mu L}u) + (\bar{d}\gamma_{\mu L}u)(\bar{u}\gamma_{\mu L}s) + 2(\bar{d}\gamma_{\mu L}s)[(\bar{d}\gamma_{\mu L}d) + (\bar{s}\gamma_{\mu L}s)] , \tag{1}$$

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$$\mathcal{O}_3=(\bar{d}\gamma_{\mu L}s)(\bar{u}\gamma_{\mu L}u)+(\bar{d}\gamma_{\mu L}u)(\bar{u}\gamma_{\mu L}s)+(\bar{d}\gamma_{\mu L}s)[2(\bar{d}\gamma_{\mu L}d)-3(\bar{s}\gamma_{\mu L}s)]\ ,$$

$$\mathcal{O}_4=(\bar{d}\gamma_{\mu L}s)(\bar{u}\gamma_{\mu L}u)+(\bar{d}\gamma_{\mu L}u)(\bar{u}\gamma_{\mu L}s)-(\bar{d}\gamma_{\mu L}s)(\bar{d}\gamma_{\mu L}d)\ ,$$

$$\mathcal{O}_5=(\bar{d}\gamma_{\mu L}t^I s)\sum_{q=u,d,s}(\bar{q}\gamma_{\mu R}t^I q)\ ,$$

$$\mathcal{O}_6=(\bar{d}\gamma_{\mu L}s)\sum_{q=u,d,s}(\bar{q}\gamma_{\mu R}q)\ , \tag{1cont'd}$$

where

$$\gamma_{\mu L,R}=\gamma_\mu\tfrac{1}{2}(1\mp\gamma_5)\ , \tag{2}$$

and t^I are the generators of $SU(3)_{\mathrm{colour}}$. The coefficients of these operators are perturbatively calculable (though this remains to be done within the staggered fermion regularization). All of these operators mediate transitions which change isospin by $\frac{1}{2}$, except for $\mathcal{O}_4$ which is a $\Delta I=\frac{3}{2}$ operator. An explanation of the $\Delta I=\frac{1}{2}$ selection rule requires one or more of the $\mathcal{O}_i$, $i\neq 4$ to be enhanced with respect to $\mathcal{O}_4$.

By using the methods of chiral perturbation theory, the matrix element of interest, namely $\langle\pi^+\pi^-|\mathcal{O}_i|K^0\rangle$, can be related to the more simple matrix elements $\langle\pi^+|\mathcal{O}_i|K^+\rangle$ and $\langle 0|\mathcal{O}_i|K^0\rangle$ [8]. Although our methods can be applied to both, in what follows we will restrict our attention to $\langle\pi^+|\mathcal{O}_i|K^+)\rangle$, which, in a general lattice theory, is related to the large time amplitude of the correlator

$$C_i(t_\pi, t_K)=\sum_{x_\pi, x_K}\langle\pi(x_\pi)\mathcal{O}_i(0)K(x_K)\rangle\ . \tag{3}$$

Here $\mathcal{O}_i$, π and K are lattice operators with the required quantum numbers, for example $K(x)=\bar{s}(x)\gamma_5 u(x)$. After integrating out the fermion fields, we are left with two types of quark propagator combinations to average over gauge configurations. These have come to be known as "eights" and "eyes" after their graphical representation. For example, the Wick expansion of $\mathcal{O}_1$ results in the graphs of fig. 1. Note that figs. 1a, 1d are products of two traces, whereas figs. 1b, 1c have only one trace.

The problem with staggered fermions is that in order to get an operator $\mathcal{O}_i$ with the right quantum numbers, we need a combination of fields distributed throughout the hypercube at the origin. We then require quark propagators from each of these sites before we can evaluate the "eight" graphs (the "eyes" are even more difficult). However, it is possible to exploit the extra Susskind flavour degrees of freedom to project out the desired operator structure from a completely local source [9], as we describe below.

3. "Eight" graphs. We introduce three staggered fermion fields χ_q, $q=$u, d, s, so that the fermionic part of the action is

$$S_F=\sum_q\sum_x\sum_\mu\tfrac{1}{2}\alpha_\mu(x)[\bar{\chi}_q(x)U_\mu(x)\chi_q(x+\mu)-\bar{\chi}_q(x+\mu)U_\mu^\dagger(x)\chi_q(x)]+\sum_q\sum_x m_q\bar{\chi}_q(x)\chi_q(x)$$

$$=\sum_q\bar{\chi}_q[\not{D}+m_q]\chi_q\ , \tag{4}$$

where $\alpha_\mu(x)=(-)^{x_1+\ldots+x_{\mu-1}}$. Their associated coordinate space "quark" fields [6,10] are defined on hypercubes at y by

$$q^{\alpha a}(y)=\frac{1}{8}\sum_A\gamma_A^{\alpha a}\chi_q(y+A)\ ,\quad \gamma_A=\gamma_1^{A_1}\gamma_2^{A_2}\gamma_3^{A_3}\gamma_4^{A_4}\ , \tag{5}$$

where $A\in\{0,1\}^4$ (i.e., $A_\mu=0$ or 1) and all components of y are even. Greek indices are to be interpreted as

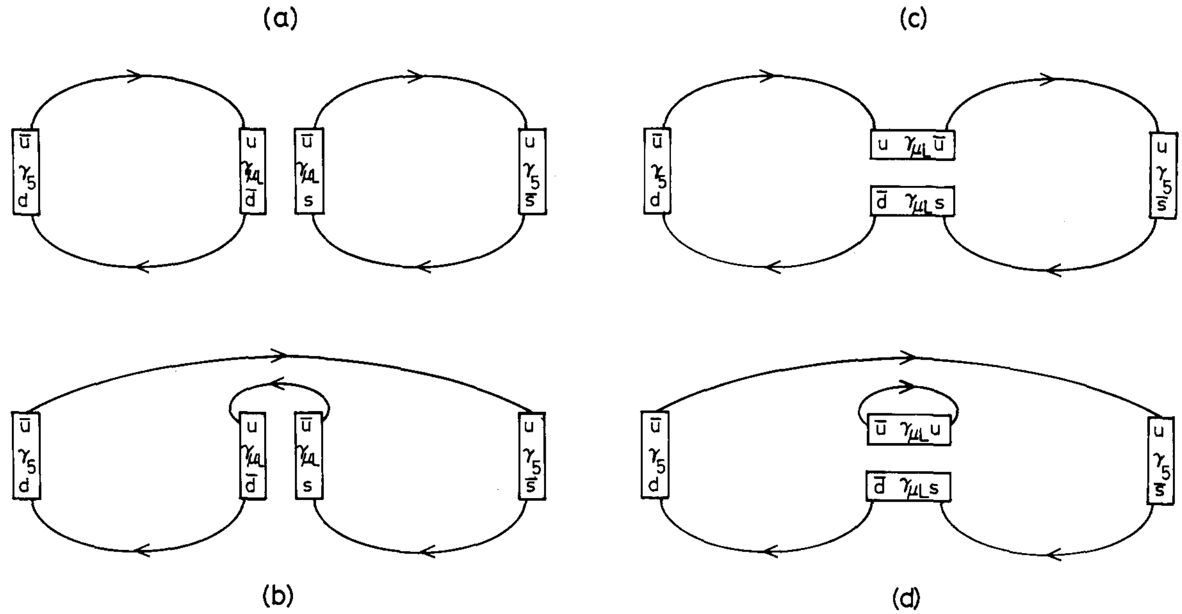

Fig. 1. Graphs obtained in the Wick expansion of $C_1(t_\pi, t_K)$. Lines correspond to quark propagators in a background gauge configuration, and loops are traced around.

spin, latin as Susskind flavour, and the lattice spacing is taken to be 1. Then we can take our meson operators to be

$$\pi_C(y) = \bar{u}(y)\gamma_5 \otimes \gamma_C^* d(y)$$

$$\text{spin} \updownarrow \quad \updownarrow \text{flavour} \tag{6}$$

$$K_D(y) = \bar{s}(y)\gamma_5 \otimes \gamma_D^* u(y) \,.$$

These should be understood as the equivalent combinations of χ-fields with gauge links introduced in a symmetric fashion to ensure gauge invariance. It should be noted that only π_5 (K_5) is a lattice Goldstone boson, but that the other fifteen π_C (K_D) correspond to physically equivalent pseudoscalar meson states in the continuum limit. This equivalence means that we are free to choose the values C and D in each graph of the Wick expansion, so that the physically relevant matrix elements can be picked out from a local source.

To see how this works out, consider the graph in fig. 1a, at first with QCD switched off. Using the four-quark operator

$$\bar{\chi}_d\chi_u(0)\bar{\chi}_u\chi_s(0) = \sum_{A,B} [\bar{d}\gamma_A \otimes \gamma_A^* u](0)[\bar{u}\gamma_B \otimes \gamma_B^* s](0) \,, \tag{7}$$

this graph, which has the two trace form, is

$$\sum_{y_\pi, y_K} \sum_{A,B} \mathrm{tr}\{S_u(0, y_\pi)\gamma_5 \otimes \gamma_C^* S_d(y_\pi, 0)\gamma_A \otimes \gamma_A^*\} \mathrm{tr}\{S_s(0, y_K)\gamma_5 \otimes \gamma_D^* S_u(y_K, 0)\gamma_B \otimes \gamma_B^*\} \,. \tag{8}$$

Here, S_q is the free quark field propagator, which is diagonal in Susskind flavour up to terms of $O(a)$ [10], and hence the trace will project out from our local four-quark operator that part with $A=C$ and $B=D$. When interactions are reintroduced we expect this projection to go through in the same manner, since Susskind flavour symmetry is present in the continuum limit. Precisely the same method can be used for the one trace type of "eight" graph (fig. 1c) if we bring it into two trace form (with respect to spin and Susskind flavour) by a Fierz transformation, though the colour factors are more complicated

4. "Eye" graphs. For the "eye" diagrams of figs. 1b, 1d this simple projection method of section 3 will not work. There are two obstacles to overcome. Firstly there is the apparent necessity of calculating the propagator $S_u(y_\pi, y_K)$, a task requiring a number of additional inversions equal to the volume of the lattice. Secondly, in either the one or the two trace forms, the correct spin structure cannot be projected out from a local four-quark operator. These two problems can be solved simultaneously. At first we will describe a method that requires two additional inversions (over and above the standard single source propagator calculation) for each pair (t_π, t_K). Afterwards we will explain how this can be reduced to three inversions in total.

Consider the one trace "eye" graph, fig. 1b. For the external mesons we choose π_5 and K_5 [see (6)], which are lattice Goldstone bosons. The four-quark operator we take is

$$\sum_A [\bar{d}\gamma_A \otimes \gamma_A^* u](0)[\bar{u}\gamma_B \otimes \gamma_C^* s](0) . \tag{9}$$

Fig. 1b then corresponds to the correlator

$$\sum_{y_\pi, y_K, A} \mathrm{tr}\{S_u(y_K, y_\pi)\gamma_5 \otimes \gamma_5^* S_d(y_\pi, 0)\gamma_A \otimes \gamma_A^* S_u(0,0)\gamma_B \otimes \gamma_C^* S_s(0, y_K,)\gamma_5 \otimes \gamma_5^*\} , \tag{10}$$

where the trace is over spin, Susskind flavour and colour (although we have suppressed the colour dependence). Once more we appeal to the fact that the quark propagators become diagonal in Susskind flavour near the continuum limit, so that the trace will pick out the $A=C$ contribution. Hence, by our choice of B and C we can uniquely specify the spin structure of the four-quark operator which survives in the continuum limit.

In terms of χ-field propagators, $G_q(x, x')$, (10) becomes

$$C(t_\pi, t_K) = \mathrm{const.} \sum_{D,E} \mathrm{tr}\{\gamma_D^\dagger \gamma_B \gamma_E \gamma_C^\dagger\} \mathrm{tr_c}\{G_u(0, D)U(D, E)G_{\pi,K}(E, 0)\} , \tag{11}$$

where $\mathrm{tr_c}$ is the trace over colour, $U(D, E)$ is the symmetric combination of gauge links between sites D and E tactily assumed in (10), and

$$G_{\pi(t_\pi), K(t_K)}(x, 0) = \sum_{y_\pi, y_K, F, G} G_s(x, y_K + F)\epsilon(F)G_u(y_K + F, y_\pi + G)\epsilon(G)G_d(y_\pi + G, 0), \tag{12}$$

with $\epsilon(x) = (-)^{x_1 + x_2 + x_3 + x_4}$ arising from our choice of mesons. Eq. (12) may be regarded as the χ-field propagator in the presence of a pion and a kaon source, and it is this particular combination of propagators which we need to calculate, rather than the general propagator $G_q(x, x')$.

To calculate (12) we require two additional inversions. First we have to solve

$$[(\not{D} + m_u)G_{\pi(t_\pi)}](x, 0) = \delta_{x_4, t_\pi}\epsilon(x)G_d(x, 0) , \tag{13}$$

and then

$$[(\not{D} + m_s)G_{\pi(t_\pi), K(t_K)}](x, 0) = \delta_{x_4, t_K}\epsilon(x)G_{\pi(t_\pi)}(x, 0) . \tag{14}$$

In these equations, $(\not{D} + m_q)$ is just the matrix in the staggered fermion action (4), so that these inversions require only minor modifications of the standard programs used in hadron mass calculations. Methods similar to this have been given elsewhere, sometimes under the name of "exponentiation" [11,12].

We have now reached the stage where we can calculate the correlator (10) corresponding to the one trace "eye" diagram with two extra inversions for each pair (t_π,t_K). The same is true for the two trace graph if we use a Fierz transformation to bring it into one trace form. However, this is still a formidable task. We can reduce the number of extra inversions required to a total of three (the standard propagator calculation together with two extra inversions) by considering the more technical question of the relation of the correlator to the matrix element $\langle \pi^+ | \mathcal{O}_i | K^+ \rangle$.

For large times, on an infinite lattice, the behaviour of the correlator (3) is dominated by the lowest mass states. On a finite lattice of course, the time cannot be too large, by there should be a region $\mathscr{R}$ in which the higher mass excitations can be neglected. For timeslices in $\mathscr{R}$ the behaviour of (3) will typically be (see refs. [5, 11])

$$C_i(t_\pi, t_K) \approx \sqrt{Z_\pi Z_K}\, A_i \exp(-m_\pi |t_\pi|) \exp(-m_K |t_K|)\,, \tag{15}$$

where $\sqrt{Z_{\pi,K}}$ are renormalization constants which can be determined in an ordinary hadron mass calculation, and m_π and m_K are the lattice pion and kaon masses. The matrix element of interest is $A_i = \langle \pi^+ | \mathcal{O}_i | K^+ \rangle$, though this will require renormalization to relate it to the physical (continuum) matrix element. In (15) we have neglected the problem of the mixing of the four-quark operators with two-quark operators (see e.g. ref. [5]). To take proper account of this it is necessary to calculate the amplitude $\langle 0 | \mathcal{O}_i | K^0 \rangle$, a task which can be tackled using methods similar to those outlined here. Now that we have the form of the correlator in $\mathscr{R}$ it is convenient to sum (15) over all $t_\pi, t_K \in \mathscr{R}$ and fit the summed correlator to this in order to extract A_i. This means that we can sum (13) and (14) over times in R, and hence we require only two extra inversions.

5. Concluding remarks. We propose to test the methods outlined in this paper using gauge configurations generated on a $16^3 \times 24$ lattice with β values in the range 6.0–6.3. Staggered fermion propagators from a single source have been generated on these configurations, at several quark mass values, for hadron mass calculations [13]. Dirichlet boundary conditions are used in the time direction, and experience shows that the region $\mathscr{R}$ for this lattice corresponds to time-slices 7–19 (the source is on timeslice 5).

To begin with, we will concentrate on comparing $\mathcal{O}_3$ with $\mathcal{O}_4$. This has the advantage that there is no possibility of mixing with two quark operators, so that only $\langle \pi^+ | \mathcal{O}_i | K^+ \rangle$ has to calculated. This is because these operators belong to the **27** representation of $SU(3)_{\text{flavour}}$. However, it is quite likely that $\Delta I = \frac{1}{2}$ enhancement is due to the other operators, all of which fall into **8** representations of $SU(3)_{\text{flavour}}$.

It should be noted that at several points our method relies on the degree to which Susskind flavour symmetry is good on the lattice. For example, the method for projecting out the correct four-quark operator rests entirely on the assumption that quark propagators are approximately diagonal in Susskind flavour space near the continuum limit. Also, at non-zero lattice spacing, the pseudoscalar masses appearing in (15) depend upon which external meson state we choose [i.e., which values of C and D we take in (6)]. Hadron mass calculations using the Edinburgh configurations [13] indicate that, on the basis of the masses of the pseudoscalar meson multiplet, Susskind flavour symmetry is respected to the order of 10% at $\beta = 6.0$, and to the order of 3–4% at $\beta = 6.15$. This gives us a reasonable change of extracting order of magnitude estimates from our data.

As emphasized in the introduction, the method proposed here is valid in the quenched approximation only. In the full theory, with dynamical fermions, it is not possible to take account of the overcounting resulting from having four-fermion species for each physical quark. In this case, it is presumably necessary to use one species of staggered fermion field, with the flavour degeneracy lifted by some generalized mass term (see for example ref. [14]). Susskind flavour would then be associated with physical flavour, but the freedom to project out spin structures would be lost.

Preliminary results using the method of ref. [5] have recently been published [15].

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